Exercise 1. Consider the function

$$u(x)=\frac{1}{(1+x^2)^{\frac{\alpha}{2}}\log\left(2+x^2\right)}\qquad x\in\mathbb{R},$$

where $0<\alpha<1$. Show that $u\in \mathrm{W}^{1,p}(\mathbb{R})$ for all $\dfrac{1}{\alpha}\leq p\leq \infty$ but that $u\notin \mathrm{L}^q(\mathbb{R})$ for all $0<q<\dfrac{1}{\alpha}$.

Exercise 2. Let $-\infty<a<b<\infty$ and $\mathrm{AC}([a,b])$ be the set of absolutely continuous functions on $[a,b]$. Recall that $u\in \mathrm{AC}([a,b])$ if and only if for all $\varepsilon>0$, there exists $\delta>0$ such that, for all $a\leq a_1<b_1<a_2<b_2<\cdots<a_m<b_m\leq b$, we have

$$\sum_{i=1}^m|b_i-a_i|<\delta \implies \sum_{i=1}^m|u(b_i)-u(a_i)|<\varepsilon.$$

Show that $\mathrm{W}^{1,1}(]a,b[)\subset \mathrm{AC}([a,b])\subset C^0([a,b])$.

Remarque 1. The first inclusion is actually an inequality, but this is harder to prove.

Exercise 3.

1. Let $u\in \mathrm{W}^{1,p}(]0,1[)$ with $1<p<\infty$. Show that if $u(0)=0$, then $x\mapsto \dfrac{u(x)}{x}\in \mathrm{L}^p(]0,1[)$ and that
$$\int_0^1\frac{|u(x)|^p}{|x|^p}dx\leq\left(\frac{p}{p-1}\right)^p\int_0^1|u'(x)|^pdx.$$
2. Conversely, assume that $u\in \mathrm{W}^{1,p}(]0,1[)$ ith $1\leq p<\infty$ and that $x\mapsto\dfrac{u(x)}{x}\in \mathrm{L}^p(]0,1[)$. Show that $u(0)=0$.
3. Let $u(x)=\dfrac{1}{1+\log\left(\frac{1}{x}\right)}$. Check that $u\in \mathrm{W}^{1,1}(]0,1[)$, $u(0)=0$, but $x\mapsto\dfrac{u(x)}{x}\notin \mathrm{L}^1(]0,1[)$.

Exercise 4. Let $\Omega\subset\mathbb{R}^d$ be an open subset. For all $u\in \mathrm{L}^1(\Omega)$, we define the variation of u in Ω by

$$V(u,\Omega)=\sup\left\{\int_\Omega u\,\mathrm{div}(\varphi)\,dx:\varphi\in C^1_c(\Omega,\mathbb{R}^d),\|\varphi\|_{\mathrm{L}^\infty(\Omega)}\leq 1\right\}.$$

The space of functions of bounded variation (BV functions) is defined by

$$\mathrm{BV}(\Omega)=\mathrm{L}^1(\Omega)\cap\{u:V(u,\Omega)<\infty\}.$$

1. Show that $\mathrm{W}^{1,1}(\Omega)\subset\mathrm{BV}(\Omega)$.
2. Let
$$H(x)=\begin{cases}1 & \text{for all }\ 0<x<1\\ 0 & \text{for all }\ -1<x\leq 0.\end{cases}$$
Show that $H\in\mathrm{BV}(]-1,1[)$ but that $H\notin \mathrm{W}^{1,1}(]-1,1[)$.
3. Let
$$u(x)=\begin{cases}x\sin\left(\dfrac{\pi}{4x}\right) & \text{for all }\ 0<x\leq 1\\ 0 & \text{for all }\ -1\leq x\leq 0.\end{cases}$$
Show that $u\in C^0([-1,1])$ but that $u\notin\mathrm{BV}(]-1,1[)$.

Remarque 2. This space can be used in an alternative construction of a solution of the problem of Plateau (it is particularly suitable in higher dimension).